An Application of Conformal Mapping to the Determination of the Characteristic Impedance of a Class of Coaxial Systems

P. A. A. LAURA AND L. E. LUISONI

Abstract—Numerical results for the characteristic impedance of the coaxial system consisting of a regular polygon concentric with a circle are presented. In the case of a square concentric with a circle the results are in excellent agreement with those obtained by Riblet [1].

Introduction

Riblet [1] has developed an ingenious approach for the accurate determination of the characteristic impedance of the coaxial system consisting of a square concentric with a circle. It has already been shown that an alternate, simpler procedure is also possible [2]. Consider the case of a polygon of regular polygonal shape of s axes of symmetry.

The mapping function which maps the interior of this polygon in the w plane onto a unit circle in the ξ plane is given by the functional relation [see Fig. 1(a)]

$$w = a_p \cdot A_s \cdot \int_0^{\xi} \frac{du}{(1 - u^s)^{2/s}} \quad (1)$$

where a_p is the apothem of the polygon and A_s is a constant [2].

Expressing (1) in series form, one has

$$w = a_p \cdot A_s \sum_{j=0}^{\infty} a_j \xi^{js+1} \quad (2a)$$

where

$$a_j = \binom{-2/s}{j} \frac{1}{js + 1} (-1)^j \quad (2b)$$

and s is the number of axes of symmetry of the configuration.

In the case of a square ($s = 4$), (2a) and (2b) yield

$$w = a_p(1.078\xi - 0.1078\xi^5 + 0.045\xi^9 - 0.026\xi^{13} + \cdots). \quad (3)$$

In this case $A_4 = 1.078$.

Now take $|\xi| \ll 1$. Obviously, the first term of (3) will predominate. Consequently, a circle of radius $r \ll 1$ will map into an approximate circle of radius $R = 1.078 a_p r$ (for $r = 0.10$ the maximum deviations with respect to a perfect circle of radius $0.1078\ a_p$ are of the order of $10^{-6}\ a_p$).

If R is the radius of the inner boundary in the w plane, the corresponding value of the polar coordinate in the ξ plane is given to a good degree of approximation by

$$r_i = \frac{R}{a_p \cdot A_s}. \quad (4)$$

A similar approach is valid if the mapping function for the infinite plane with a polygonal hole is known [Fig. 2(a)].

Now the mapping function is given by a functional relation of the form

$$w = a_p \cdot A_s' \sum_{n=0}^{\infty} a_{1-ns} \xi^{-1ns}, \qquad a_1 = 1 \quad (5)$$

where s is the number of axes of symmetry of the configuration.

Now take $r = r_0 \gg 1$. The first term will again predominate

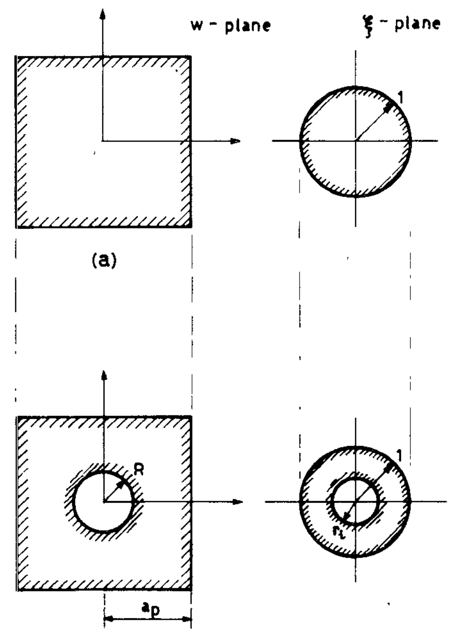

Fig. 1. Approximate conformal mapping of a doubly connected region (case A).

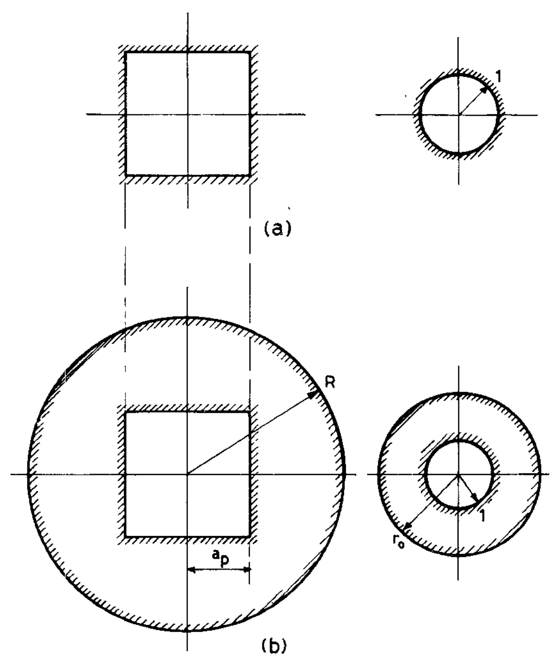

Fig. 2. Approximate conformal mapping of a doubly connected region (case B).

since the remaining involve negative powers of the radial variable. Consequently, the radius of the approximate circle in the w plane is given by (Fig. 2)

$$R = A_s' \cdot r_0 \cdot a_p. \quad (6)$$

Since in both cases the domains can be transformed onto approximate annular regions, the expressions for the characteristic impedances are as follows:[1]

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P. A. A. Laura is with the Institute of Applied Mechanics, Puerto Belgrano Naval Base, Argentina, and the Universidad Nacional de Sur, Bahía Blanca, Argentina.
L. E. Luisoni is with the Institute of Applied Mechanics, Puerto Belgrano Naval Base, Argentina.

[1] The numerical values are obtained here for the case of vacuum.

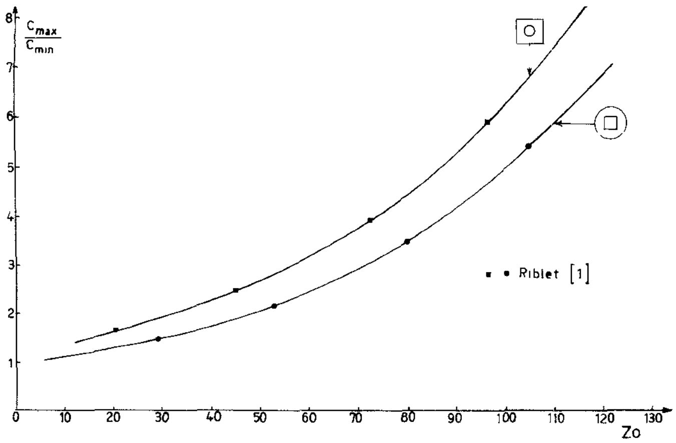

Fig. 3. Characteristic impedance of a coaxial system when one of the boundaries is a square: comparison of results.

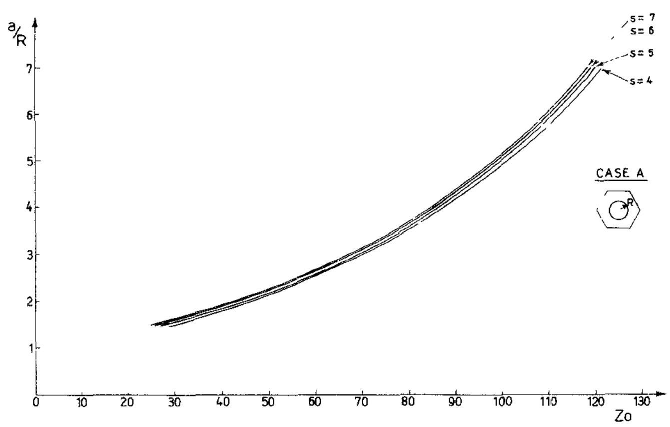

Fig. 4. Characteristic impedance of a coaxial system (case A).

Case A (Fig. 1):

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \ln \frac{1}{r_i} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \left(\ln A_s + \ln \frac{a_p}{R}\right). \quad (7)$$

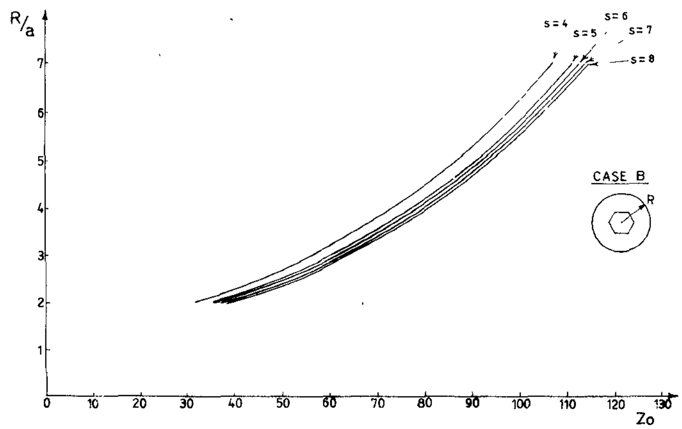

Fig. 5. Characteristic impedance of a coaxial system (case B).

Case B (Fig. 2):

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \ln r_{0'} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \left(-\ln A_s' + \ln \frac{R}{a_p}\right). \quad (8)$$

Values of A_s and A_s' are given in [2].

Numerical Results

Fig. 3 shows a comparison between those obtained by Riblet [1] and the ones obtained using expressions (7) and (8) for a square outer (inner) shape.[2] The agreement is excellent in all cases.

Figs. 4 and 5 depict results for different polygonal configurations.

It must be emphasized that Riblet's and present values are theoretical.

References

[1] H. J. Riblet, "An accurate determination of the characteristic impedance of the coaxial system consisting of a square concentric with a circle," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 714–715, Aug. 1975.
[2] P. A. A. Laura and L. E. Luisoni, "Approximate determination of the characteristic impedance of the coaxial system consisting of a regular polygon concentric with a circle," this issue, pp. 160–161.

[2] C_{max}: length of the outer boundary. C_{min}: length of the inner boundary.